TRINITY COLLEGE

ADMISSIONS QUIZ (MATHEMATICS)

SPECIMEN TEST 1

There are ten questions below which are on various areas of mathematics. They are of varying levels of difficulty: some may be easy and others could be hard. You are not expected to answer all of them, or necessarily to give complete answers to questions. You should just attempt those that appeal to you, and they will be used as a basis for discussion in the interview that follows. You should bring what you have written with you to the interview.

1. In a tennis tournament there are $2n$ participants. In the first round of the tournament, each player plays exactly once, so there are n games. Show that the pairings for the first round can be arranged in exactly $(2n-1)!/2^{n-1}(n-1)!$ ways.

2. Let L_1 and L_2 be two lines in the plane, with equations $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively. Suppose that they intersect at an acute angle θ. Show that

$$\tan(\theta) = \left| \frac{m_1 - m_2}{1 + m_1m_2} \right| .$$

3. Calculate $\int_0^{\pi} (x \sin x)^2 \, dx$.

4. Of the numbers $1, 2, 3, \ldots, 6000$, how many are not multiples of $2, 3$ or 5?

5. There is a pile of 129 coins on a table, all unbiased except for one which has heads on both sides. Bob chooses a coin at random and tosses it eight times. The coin comes up heads every time. What is the probability that it will come up heads the ninth time as well?

6. A packing case is held on the side of a hill and given a kick down the hill. The hill makes an angle of θ to the horizontal, and the coefficient of friction between the packing case and the ground is μ. What relationship between μ and θ guarantees that the packing case eventually comes to rest? Let gravitational acceleration be g. If the relationship above is satisfied, what must the initial speed of the packing case be to ensure that the distance it goes before stopping is d?

[Please turn over]

7. Let $\binom{n}{r}$ stand for the number of subsets of size r taken from a set of size n. (This is the number of ways of choosing r objects from n if the order of choice does not matter. You may be more familiar with the notation ${}^nC_r$, in which case feel free to use it.) Every subset of the set $\{1, 2, \ldots, n\}$ either contains the element 1 or it doesn't. By considering these two possibilities, show that

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}.$$

By using a similar method, or otherwise, prove that

$$\binom{n-2}{r-2} + 2\binom{n-2}{r-1} + \binom{n-2}{r} = \binom{n}{r}.$$

8. One end of a rod of uniform density is attached to the ceiling in such a way that the rod can swing about freely with no resistance. The other end of the rod is held still so that it touches the ceiling as well. Then the second end is released. If the length of the rod is l metres and gravitational acceleration is g metres per second squared, how fast is the unattached end of the rod moving when the rod is first vertical?

9. Let M be a large real number. Explain briefly why there must be exactly one root w of the equation $Mx = e^x$ with $w > 1$. Why is $\log M$ a reasonable approximation to w? Write $w = \log M + y$. Can you give an approximation to y, and hence improve on $\log M$ as an approximation to w?

10. Twenty balls are placed in an urn. Five are red, five green, five yellow and five blue. Three balls are drawn from the urn at random without replacement. Write down expressions for the probabilities of the following events. (You need not calculate their numerical values.)

(i) Exactly one of the balls drawn is red.

(ii) The three balls drawn have different colours.

(iii) The number of blue balls drawn is strictly greater than the number of yellow balls drawn.